Sec. 5.2 #8 | MATH 2107 Assignment #4

$$A=\begin{bmatrix}1&1&-1&0&2\\-2&0&2&4&4\\2&2&-2&0&1\\-3&-1&3&4&5\end{bmatrix}\sim\begin{bmatrix}1&1&-1&0&2\\0&2&0&4&8\\0&0&0&0&-3\\0&2&0&4&11\end{bmatrix}$$

$$\sim\begin{bmatrix}1&1&-1&0&2\\0&1&0&2&4\\0&0&0&0&1\\0&2&0&4&11\end{bmatrix}\sim\begin{bmatrix}1&1&-1&0&2\\0&1&0&2&4\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix}$$

pivot columns.

A basis for Col A is $\left\{\underbrace{\begin{bmatrix}1\\-2\\2\\-3\end{bmatrix}}_{u},\underbrace{\begin{bmatrix}1\\0\\2\\-1\end{bmatrix}}_{v},\underbrace{\begin{bmatrix}2\\4\\1\\5\end{bmatrix}}_{z}\right\}$.

$$A^T=\begin{bmatrix}1&-2&2&-3\\1&0&2&-1\\-1&2&-2&3\\0&4&0&4\\2&4&1&5\end{bmatrix}\sim\begin{bmatrix}1&-2&2&-3\\0&2&0&2\\0&0&0&0\\0&4&0&4\\0&8&-3&11\end{bmatrix}\sim\begin{bmatrix}1&-2&2&-3\\0&1&0&1\\0&8&-3&11\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

$$\sim\begin{bmatrix}1&-2&2&-3\\0&1&0&1\\0&0&-3&3\\0&0&0&0\\0&0&0&0\end{bmatrix}\sim\begin{bmatrix}1&-2&2&-3\\0&1&0&1\\0&0&1&-1\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

$$A^Tx=0 \Rightarrow x=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}-x_4\\-x_4\\x_4\\x_4\end{bmatrix}.$$

A basis for null(A^T) is $\left\{\underbrace{\begin{bmatrix}-1\\-1\\1\\1\end{bmatrix}}_{w}\right\}$.

[illegible]

To show that every vector in col(A) is orthogonal to every vector in null (AT), it is enough to show that every vector in a basis for col(A) is orthogonal to every vector in a basis for Null(AT).

$u \cdot w = -1+2+2-3 = 0$

$v \cdot w = -1+0+2-1 = 0$

$z \cdot w = -2-4+1+5 = 0$

$\Rightarrow (\mathrm{col}(A))^{\perp} = \mathrm{null}(A^T)$.

Sec. 5.2 #18

$$\mathrm{proj}_W v = \frac{v \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{v \cdot u_2}{u_2 \cdot u_2} u_2 + \frac{v \cdot u_3}{u_3 \cdot u_3} u_3$$

$$= \frac{1}{2}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} + \frac{-1}{4}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

$$= \begin{bmatrix}\frac{1}{2}-\frac{1}{4}+0\\ \frac{1}{2}+\frac{1}{4}+0\\ 0+\frac{1}{4}+\frac{1}{2}\\ 0-\frac{1}{4}+\frac{1}{2}\end{bmatrix} = \begin{bmatrix}\frac{1}{4}\\ \frac{3}{4}\\ \frac{3}{4}\\ \frac{1}{4}\end{bmatrix} = \frac{1}{4}\begin{bmatrix}1\\3\\3\\1\end{bmatrix}$$

$\boxed{\text{Sec. 5.3, \# 10}}$

let $u_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 5 \end{bmatrix}$, $u_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$.

Take $v_1 = u_1$

$$v_2 = u_2 - \text{proj}'_{v_1} u_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 5 \end{bmatrix} - \frac{1-1-1+5}{4} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 2 \\ 4 \end{bmatrix}$$

$$v_3 = u_3 - \text{proj}'_{v_1} u_3 - \text{proj}'_{v_2} u_3$$

$$= \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} - \frac{4}{4} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} - \frac{0}{4+4+16} v_2$$

$$\boxed{A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ -1 & 1 & 0 \\ 1 & 5 & 1 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

An orthogonal basis for Col(A) is

$$\left\{ v_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ -2 \\ 2 \\ 4 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

$$q_1 = \frac{v_1}{\|v_1\|} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$q_2 = \frac{v_2}{\|v_2\|} = \frac{1}{2\sqrt{6}}\begin{bmatrix} 0 \\ -2 \\ 2 \\ 4 \end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix}$$

$$q_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{2}{\sqrt{6}} & 0 \end{bmatrix}.$$

$$A = QR \Rightarrow R = Q^T A$$

$$R = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -1/2 & 1/2 \\ 0 & -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ -1 & 1 & 0 \\ 1 & 5 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 12/\sqrt{6} & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}$$

$\boxed{\text{Sec. 5.4, \#6}}$

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 3 & 2 & 4 \\ 0 & 4 & 2 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 & 0 \\ 3 & 2-\lambda & 4 \\ 0 & 4 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)\begin{vmatrix} 2-\lambda & 4 \\ 4 & 2-\lambda \end{vmatrix} - 3\begin{vmatrix} 3 & 4 \\ 0 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)(\lambda^2 - 4\lambda + 4 - 16) - 3(6 - 3\lambda)$$

$$= (2-\lambda)(\lambda^2 - 4\lambda - 12) - 9(2-\lambda)$$

$$= (2-\lambda)(\lambda^2 - 4\lambda - 12 - 9)$$

$$= (2-\lambda)(\lambda^2 - 4\lambda - 21) = (2-\lambda)(\lambda - 7)(\lambda + 3)$$

$$\det(A - \lambda I) = 0 \iff \lambda = 2, 7, -3$$

$\underline{\text{If } \lambda = 2}$: $A - 2I = \begin{bmatrix} 0 & 3 & 0 \\ 3 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$(A - 2I)x = 0 \iff x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -(4/3)x_3 \\ 0 \\ x_3 \end{bmatrix} = \frac{1}{3}x_3 \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix}.$$

$\underline{\underline{\lambda = 7}}$: $A - 7I = \begin{bmatrix} -5 & 3 & 0 \\ 3 & -5 & 4 \\ 0 & 4 & -5 \end{bmatrix} \sim \begin{bmatrix} -15 & 9 & 0 \\ 15 & -25 & 20 \\ 0 & 4 & -5 \end{bmatrix}$

$$\sim \begin{bmatrix} -15 & 9 & 0 \\ 0 & -16 & 20 \\ 0 & 4 & -5 \end{bmatrix} \sim \begin{bmatrix} -15 & 9 & 0 \\ 0 & -4 & 5 \\ 0 & 4 & -5 \end{bmatrix} \sim \begin{bmatrix} -5 & 3 & 0 \\ 0 & -4 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A - 7I)x = 0 \iff x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (3/4)x_3 \\ \frac{5}{4}x_3 \\ x_3 \end{bmatrix} = \frac{1}{4}x_3 \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}$$

$\lambda = -3$:

$$A+3I = \begin{bmatrix} 5 & 3 & 0 \\ 3 & 5 & 4 \\ 0 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 15 & 9 & 0 \\ -15 & -25 & -20 \\ 0 & 4 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 15 & 9 & 0 \\ 0 & -16 & -20 \\ 0 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 15 & 9 & 0 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 5 & 3 & 0 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A+3I)x = 0 \Leftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{4}x_3 \\ -\frac{5}{4}x_3 \\ x_3 \end{bmatrix} = \frac{1}{4}x_3 \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}.$$

$$\lambda_1 = 2 \Rightarrow v_1 = \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix}$$

$$\lambda_2 = 7 \Rightarrow v_2 = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}$$

$$\lambda_3 = -3 \Rightarrow v_3 = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}$$

Since A is a 3×3 symmetric matrix and ~~eigenvalues~~ it has three distinct eigenvalues, then corresponding eigenvectors v_1, v_2, v_3 are orthogonal.

$$q_1 = \frac{1}{\|v_1\|} v_1 = \frac{1}{\sqrt{16+9}} v_1 = \frac{1}{5}\begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix}$$

$$q_2 = \frac{1}{\|v_2\|} v_2 = \frac{1}{\sqrt{50}}\begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix} = \frac{1}{5\sqrt{2}}\begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}$$

$$q_3 = \frac{1}{\|v_3\|} v_3 = \frac{1}{5\sqrt{2}} \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}.$$

$$Q = \begin{bmatrix} -4/5 & 3/5\sqrt{2} & 3/5\sqrt{2} \\ 0 & 5/5\sqrt{2} & -5/5\sqrt{2} \\ 3/5 & 4/5\sqrt{2} & 4/5\sqrt{2} \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$